

## ASSIGNMENT 4

### Reading:

105 Notes 6.1-6.2, 3.1-3.3  
Hand & Finch 3.1-3.3

1.

Generalize the Euler equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{\partial \mathcal{L}}{\partial y}$$

to the case in which  $\mathcal{L}$  is a function of  $t$ ,  $y$ ,  $\dot{y}$ , and  $\ddot{y}$ . Derive the new Euler equation for this case. Assume that  $y(t_1)$ ,  $y(t_2)$ , and  $\dot{y}(t_1)$ ,  $\dot{y}(t_2)$  are not varied, *i.e.* both the value and the slope of  $y$  are fixed at each endpoint.

[*Hint:* Compared to the derivation of the usual Euler equation, when you calculate the variation of the action  $J$  with the parameter  $\alpha$ , you will have an extra term in the integrand. Integrate that term by parts twice.]

2.

A bead moves in a constant gravitational field  $\mathbf{a} = \hat{\mathbf{x}}g$  with an initial velocity  $|\mathbf{v}| = v_0$ , where  $g$  and  $v_0$  are positive constants. It is constrained to slide along a frictionless wire which has an unknown shape  $y(x)$ . (Notice that  $\hat{\mathbf{x}}$  points down and  $\hat{\mathbf{y}}$  points to the right in this problem.)

(a)

Show that the shape  $y(x)$  which minimizes the bead's transit time between two fixed points  $(0,0)$  and  $(X,Y)$  is given by a set of parametric equations

$$\begin{aligned} x &= x_0 + a(1 - \cos \phi) \\ y &= y_0 + b(\phi - \sin \phi), \end{aligned}$$

where  $\phi$  is the parameter. This is the famous *brachistochrone problem*. The solution is a *cycloid* – the path of a dot painted on a rolling wheel.

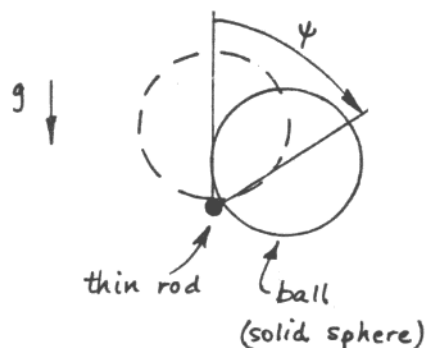
(b)

In terms of  $v_0$ ,  $g$ ,  $X$ , and  $Y$ , what are the values of the constants  $x_0$ ,  $y_0$ ,  $a$ , and  $b$  which yield the

optimal trajectory? Give definite answers where you can; otherwise provide equations which, if solved, would yield those values. [*Hint:* see Hand & Finch, problems 2.9 and 2.10.]

3.

Starting from a vertical position at rest, a solid ball resting on top of a thin rod falls off.



While in contact with the rod, the ball rolls without slipping. Using the method of Lagrange undetermined multipliers, find the angle  $\psi$  at which the ball leaves the rod ( $\psi \equiv 0$  initially).

4.

Consider a simple, plane pendulum consisting of a mass  $m$  attached to a string of length  $l$ . Only small oscillations need be considered. After the pendulum is set into motion, the length of the string is shortened at a constant rate  $dl/dt = -\alpha$ , where  $\alpha > 0$ . (The string is pulled through a small hole located at a constant position, so the pendulum's suspension point remains fixed.) Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy of the pendulum, and discuss the conservation of energy for the system.

5.

A particle of mass  $m$  and velocity  $\mathbf{v}_1$  leaves a semi-infinite space  $z < 0$ , where the potential energy is a constant  $U_1$ , and enters the remaining space  $z > 0$ , where the potential is a constant  $U_2$ .

(a)

Use symmetry arguments to find two constants of the motion.

(b)

Use these two constants to obtain the new velocity  $\mathbf{v}_2$ .

6.

The Lagrangian for a (physically interesting) system is

$$\mathcal{L}(\varphi, \dot{\varphi}, \theta, \dot{\theta}, \psi, \dot{\psi}, t) = \frac{1}{2}I(\dot{\varphi}^2 \sin^2 \theta + \dot{\theta}^2) + \frac{1}{2}I_3(\dot{\varphi} \cos \theta + \dot{\psi})^2 - mgh \cos \theta ,$$

where  $(\varphi, \theta, \psi)$  are Euler angles and  $(I, I_3, mgh)$  are constants.

(a)

Find two cyclic coordinates and obtain the two corresponding conserved canonically conjugate momenta.

(b)

Find a third constant of the motion.

(c)

Using the results of (a) and (b), express  $\dot{\theta}^2$  as a function only of  $\theta$  and constants.

7.

The interaction Lagrangian for a system consisting of a relativistic test particle of mass  $m$  and charge  $e$  moving in a static electromagnetic field is

$$\mathcal{L}(\mathbf{x}, \mathbf{v}, t) = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} + e\mathbf{v} \cdot \mathbf{A} - e\phi ,$$

where  $\mathbf{x}$  is the particle's position,  $\mathbf{v}$  is its velocity,  $\phi(\mathbf{x})$  is the electrostatic potential ( $\mathbf{E} = -\nabla\phi$ ),  $\mathbf{A}$  is the (static) magnetic vector potential ( $\mathbf{B} = \nabla \times \mathbf{A}$ ), and  $c$  is the speed of light.

(a)

Write down the canonical momenta  $(p_1, p_2, p_3)$  which are conjugate to the Cartesian coordinates

$(x_1, x_2, x_3)$ .

(b)

Compute the Hamiltonian  $\mathcal{H}(\mathbf{x}, \mathbf{v}, t)$ .

(c)

Re-express  $\mathcal{H}(\mathbf{x}, \mathbf{v}, t)$  as the function  $\mathcal{H}(\mathbf{x}, \mathbf{p}, t)$ .

(d)

Show that  $\mathcal{H}$  is conserved. Is it equal to

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} ,$$

the total (relativistic) energy of the test particle? Explain.

8.

Consider  $f$  and  $g$  to be any two continuous functions of the generalized coordinates  $q_i$  and canonically conjugate momenta  $p_i$ , as well as time:

$$f = f(q_i, p_i, t) \\ g = g(q_i, p_i, t) .$$

The Poisson bracket of  $f$  and  $g$  is defined by

$$[f, g] \equiv \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} ,$$

where summation over  $i$  is implied. Prove the following properties of the Poisson bracket:

(a)

$$\frac{df}{dt} = [f, \mathcal{H}] + \frac{\partial f}{\partial t}$$

(b)

$$\dot{q}_i = [q_i, \mathcal{H}]$$

(c)

$$\dot{p}_i = [p_i, \mathcal{H}]$$

(d)

$$[p_i, p_j] = 0$$

(e)

$$[q_i, q_j] = 0$$

(f)

$$[q_i, p_j] = \delta_{ij} ,$$

where  $\mathcal{H}$  is the Hamiltonian. If the Poisson bracket of two quantities is equal to unity, the quantities are said to be *canonically conjugate*. On the other hand, if the Poisson bracket vanishes, the quantities are said to *commute*.

(g)

Show that any quantity that does not depend explicitly on the time and that commutes with the Hamiltonian is a constant of the motion.